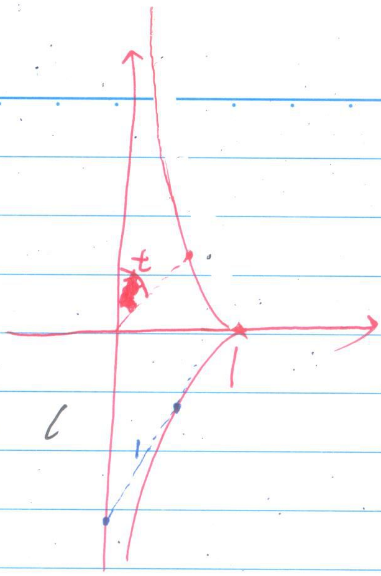
Assignment1:

(1) $\alpha(t) = (\sin t, \cos t + \log \tan\frac{t}{2})$

$\Rightarrow \alpha'(t) = (\cos t, -\sin t + \frac{1}{\sin t})$



(a) $\alpha' = 0$ for $t \in (0, \pi)$

$\Leftrightarrow t = \frac{\pi}{2}$

(b) denote the tangent line of α at $\alpha(t)$ be l

$\because \alpha' = (\cos t, -\sin t + \frac{1}{\sin t})$

$\therefore$ the intersection of l and y-axis is

$(0, \log \tan\frac{t}{2})$ for $t \neq \frac{\pi}{2}$

$\therefore$ the length of the segment of the tangent of α between the point of tangency and the y-axis is

$$\sqrt{\sin^2 t + \cos^2 t} = 1 \quad \text{for } t \neq \frac{\pi}{2}$$

- Remark:
  - if the curve is C^1 at $\alpha(\frac{\pi}{2})$ then the tangent lines of this curve is continuous w.r.t t
    $\therefore$ For $t = \frac{\pi}{2}$, the answer is true
  - if the curve is only C^0 at $\alpha(\frac{\pi}{2})$ then there is no tangent line at $\alpha(\frac{\pi}{2})$

(2) "$\Rightarrow$" $\because \langle T, u\rangle = \cos\theta_0$ is a constant

$\therefore \langle T', u\rangle = 0$ since u is a constant

$k\langle N, u\rangle = 0$

$\therefore \langle N, u\rangle = 0$ since $k \neq 0$

$\therefore u = |u|(\cos\theta_0 T + \sin\theta_0 B)$ for some θ

differentiate $\quad \because \langle T, u\rangle = \cos\theta_0 \quad \therefore \theta$ is a constant

$0 = \cos\theta_1 T' + \sin\theta_1 B'$

$= \cos\theta_1 kN - \sin\theta_1 \tau N$

$\therefore \frac{k}{\tau}$ is a constant

"$\Leftarrow$" $\because \frac{k}{\tau}$ is constant

$\therefore \frac{k}{\tau} = \frac{\sin\theta_2}{\cos\theta_2}$ for some θ_2

$\therefore (\cos\theta_2 T + \sin\theta_2 B)' = 0$

$\therefore$ let $\vec{u} = \cos\theta_2 T + \sin\theta_2 B$

then $\langle T, \vec{u}\rangle$ is constant.

(3) "$\Rightarrow$" W.L.O.G, we can assume that the sphere is centered at the origin.

$\therefore \ |\alpha|^2 = \langle \alpha, \alpha \rangle$ is a constant

$\therefore \ 0 = \langle \alpha, \alpha \rangle' = 2\langle \alpha, \alpha' \rangle = 2\langle \alpha, T \rangle \quad \Rightarrow \langle \alpha, T \rangle = 0$

$0 = \langle \alpha, \alpha \rangle'' = 2\langle \alpha', \alpha' \rangle + 2\langle \alpha, \alpha'' \rangle$

$= 2 + 2\langle \alpha, kN \rangle \quad \Rightarrow \langle \alpha, N \rangle = -\frac{1}{k}$

$0 = \langle \alpha, \alpha \rangle''' = 2\langle \alpha', kN \rangle + 2\langle \alpha, k'N + kN' \rangle$

$= 0 + 2\langle \alpha, k'N - k^2T + k\tau B \rangle$

$= 2k'\langle \alpha, N \rangle + 2k\tau\langle \alpha, B \rangle$

$= -\frac{2k'}{k} + 2k\tau\langle \alpha, B \rangle. \quad \Rightarrow \langle \alpha, B \rangle = \frac{k'}{k^2\tau}$

$\because \ \{T, N, B\}$ is an orthonormal basis of R^3

$\therefore$

$$\alpha = -\frac{1}{k}N + \frac{k'}{k^2\tau}B = -\rho N - \rho'\lambda B$$

where $\rho = \frac{1}{k}$, $\lambda = \frac{1}{\tau}$.

$\therefore \ \langle \alpha, \alpha \rangle = \rho^2 + (\rho')^2\lambda^2$ is a constant since α lies on a sphere centered at the origin.

"$\Leftarrow$" let $\beta = \alpha + \rho N + \rho'\lambda B$

then $\beta' = \alpha' + \rho' N + \rho N' + \rho''\lambda B + \rho'\lambda' B + \rho'\lambda B'$

$= T + \rho' N - \rho k T + \rho\tau B + \rho''\lambda B + \rho'\lambda' B - \rho'\lambda\tau N$

$= T + \rho' N - T + \rho\tau B + \rho''\lambda B + \rho'\lambda' B - \rho' N$

$= (\rho\tau + \rho''\lambda + \rho'\lambda')B$

$= (\frac{\rho}{\lambda} + \rho''\lambda + \rho'\lambda')B$

By assumption, $\rho^2 + (\rho')^2\lambda^2$ is a constant

$\therefore \ 0 = 2\rho\rho' + 2\rho'\rho''\lambda^2 + (\rho')^2 2\lambda\lambda'$

$\therefore \ \rho + \rho''\lambda^2 + \lambda\rho'\lambda' = 0$ since by assumption $k' \neq 0$.

$\therefore$ we have $\beta' = 0$, which means β is a fixed point

$\because \ |\beta - \alpha|^2 = \rho^2 + (\rho')^2\lambda^2$ is a constant

$\therefore \ \alpha$ lies on a sphere centered at β.

(4)(a) Suppose not. Then there are $\{S_1^j, S_2^j, S_3^j\}$ such that

$\begin{cases} \alpha(S_1^j), \alpha(S_2^j), \alpha(S_3^j) \text{ are collinear for each } j=1,2,\cdots \\ \text{and } S_1^j, S_2^j, S_3^j \to S_0 \text{ as } j\to\infty \end{cases}$

$\therefore$ for each j, there are unit vectors V^j, W^j such that

$\begin{cases} |V^j|=|W^j|=1, \quad \langle V^j, W^j\rangle=0 \\ \langle \alpha(S_i^j)-\alpha(S_2^j), V^j\rangle = \langle \alpha(S_i^j)-\alpha(S_2^j), W^j\rangle=0 \end{cases}$

each $i=1,2,3$.

$\therefore$ Consider the functions $\langle \alpha(s)-\alpha(S_2^j), V^j\rangle$, $\langle \alpha(s)-\alpha(S_2^j), W^j\rangle$

then there are $x^j\in(S_1^j, S_2^j)$, $y^j\in(S_2^j, S_3^j)$, $z^j\in(x^j, y^j)$ such that

$\begin{cases} \langle \alpha'(x^j), V^j\rangle = \langle \alpha'(y^j), V^j\rangle = \langle \alpha'(x^j), W^j\rangle = \langle \alpha'(y^j), W^j\rangle=0 \\ \langle \alpha''(z^j), V^j\rangle = \langle \alpha''(z^j), W^j\rangle = 0 \end{cases}$

$\because$ $\alpha'(x^j), \alpha'(y^j)\to\alpha'(S_0)$ as $j\to\infty$

$\therefore$ $\{V^j, W^j\}\to\{V^0, W^0\}$ where V^0, W^0 are unit vectors

and $\operatorname{span}\{V^0, W^0\} = \operatorname{span}\{N(S_0), B(S_0)\}$

$\because$ $\langle \alpha''(z^j), V^j\rangle = \langle \alpha''(z^j), W^j\rangle=0$

$\therefore$ $\langle kN(S_0), V^0\rangle = \langle kN(S_0), W^0\rangle = 0$

But since $\operatorname{span}\{V^0, W^0\} = \operatorname{span}\{N(S_0), B(S_0)\}$

$\therefore$ $k(S_0)=0$, this is a contradiction to $k(S_0)\neq 0$

$\therefore$ if $S_3>S_2>S_1$ are sufficienly close to S_0, $\alpha(S_1), \alpha(S_2), \alpha(S_3)$ are not collinear.

(b) $\because$ $\alpha(S_1), \alpha(S_2), \alpha(S_3)$ are not collinear

$\therefore$ there is a unit vector $\vec{n}$ such that

$\langle \alpha(S_i)-\alpha(S_2), \vec{n}\rangle = 0$ for each $i=1,2,3$

Similarly, there are $x\in(S_1, S_2)$, $y\in(S_2, S_3)$, $z\in(x,y)$ such that

$\begin{cases} \langle \alpha'(x), \vec{n}\rangle = \langle \alpha'(y), \vec{n}\rangle = 0 \\ \langle \alpha''(z), \vec{n}\rangle = 0 \end{cases}$

$\therefore$ $\lim_{S\to S_0}\langle T(s), \vec{n}\rangle = \lim_{S\to S_0}\langle kN(s), \vec{n}\rangle = 0$

$\because$ $k(S_0)\neq 0$

$\therefore$ $\vec{n}$ will tend to $B(S_0)$ or $-B(S_0)$

$\therefore$ the unique plane containing $\alpha(S_1), \alpha(S_2), \alpha(S_3)$ will approach to the plane spanned by $T(S_0), N(S_0)$.